Report on the outcomes of a Short-Term Scientific Mission[1]

Action number: CA20111

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<u>Details of the STSM</u>

Title: Categorical semantics for propositional dependent type theories

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Description of the work carried out during the STSM

Description of the activities carried out during the STSM. Any deviations from the initial working plan shall also be described in this section.

The STSM focused on weakenings for theories of dependent types (DTT) with respect to the strength of the computation rules (e.g.whether they hold judgementally or propositionally) for the type constructors of the DTT itself, following a categorical point of view.

Together with Benno van den Berg (host) and his PhD student Daniël Otten, we worked on categorical semantics of both propositional and strict DTTs, focusing on the established notions of semantics provided by comprehension categories, as well as with van den Berg's notion of path category. In detail we investigated how path categories provide a semantics for propositional type theories, and compare this with the established notion of semantics.

(A) We have worked on reviewing and clarifying the notion of semantics for propositional type theories (i.e. DTTs with propositional computation rules) provided by the notion of path categories. In detail, we have investigated what notion of DTT is actually modelled by a path category, focusing on the nature of the identities and on the notion of substitution.

(B) We have focused on the categories of the models of this given weak theory according to both the notions of semantics (resp. comprehension categories and path categories), and searched for a way of converting the ones into the others.

(C) We have worked on the problem of the substitution in path categories. This means that we have looked for a coherence result for path categories that repairs the issue of the substitution, when this is not strictly functorial. The problem is similar to the one for locally cartesian closed categories, for which Martin Hofmann produced a coherence theorem, and our approach has been based on Hofmann's one and other approaches that produced coherence results for DTT's in the literature (Lumsdaine, Warren, Bocquet).

(D) We have organised a workshop on weak type theories in Amsterdam on the 11th and 12th of May, together with other experts of weak DTTs, in order to share and compare our results (https://dutchcats.github.io/).

We believe that our project is being fruitful, and we are keeping on carrying it out after the STSM, with weekly communication with van den Berg and Otten. The project is expected to contribute to EuroProofNet's Working Group 6 (Type Theory) as it fits the group's aim of developing a comprehensive theory of type theories.

Description of the STSM main achievements and planned follow-up activities

Description and assessment of whether the STSM achieved its planned goals and expected outcomes, including specific contribution to Action objective and deliverables, or publications resulting from the STSM. Agreed plans for future follow-up collaborations shall also be described in this section.

I presented our results and work in progress at the Workshop on Doctrines and Fibrations (https://events.math.unipd.it/WDF2023/) on the 1st of June. Here is the YouTube recording of my talk:

https://www.youtube.com/watch?v=a6JNebaYXgU&t=400s&pp=ygUPbWF0dGVvIHNwYWRldHRv

. Our results and our work in progress are summarised here:

(A) Comparing the notion of path category to the one of comprehension category, we showed that path categories can be naturally characterised as Grothendieck fibrations carrying a structure of comprehension category with three properties: it has propositional identity types; it has strong dependent sum types; it is democratic (i.e. contexts are represented by types in the empty context). This shows that a path category is a (pseudo) model of a DTT with propositional identity types and strong dependent sum types. A pseudo model, because the obtained comprehension category in general is non split and the type constructors are only weakly stable.

We are working on proving that this correspondence is a 2-equivalence and we hope that it generalises the 2-equivalence by Clairambault and Dybier between finitely complete categories and democratic comprehension categories with extensional identities and strong sigmas.

Moreover, we know that any non-split comprehension category with weakly stable type constructors is equivalent to a split one with stable type constructors (results by Lumsdaine, Warren, Bocquet). In other words, a pseudo model of the DTT that we determined is equivalent to an actual (strict) model. This proves that any path category has a split replacement, hence it seems we have proved the coherence result for propositional identity types that we were looking for. We are working on writing all of this down in full detail.

This solves both our problem of clarifying in what sense a path category is a model of what DTT, and our problem of fixing (splitting) the substitution of a path category (what Hofmann did for lcc categories).

(B) In (A) we mentioned that we have a mapping from path categories to democratic comprehension categories with weakly stable propositional identities and strong sigmas. We believe this mapping is a 2-equivalence and we are working on proving this. This would be a final answer to our problem of comparing the two notions of semantics (path categories and comprehension categories), as it would conclude that, for a DTT with strong sigmas and propositional identities, the two classes of models provided by the two notions of semantics are equivalent.

(C) About the problem of the coherence we already answered in (A): it seems we do have a coherence result for path categories that fixes the problem of the substitution.

Other work in project: we are working on weakening the notion of path category to model propositional identites, propositional sigmas (in place of strong sigmas), and propositional pis. Then we will generalise our results of (a-b-c) to this notion. It would constitute a very general notion of semantics that allows one study via categorical tools basically every notion of DTT.