Short-Term Scientific Mission Grant
- APPLICATION FORM[1] -

Action number:

Applicant name:

Details of the STSM

Title: New axiomatization of synthetic differential geometry

Start and end date: 23/06/2024 to 29/06/2024

Goals of the STSM

Purpose and summary of the STSM.

(max.200 word)

Recent advances in synthetic algebraic geometry (SAG) suggest a revision of the axioms commonly used in synthetic differential geometry (SDG). Both synthetic approaches to geometry are very suitable for formalization in (homotopy) type theory. There is already a synthetic formalization of the notion of scheme in SAG in cubical agda and it is reasonable to expect that code can be shared between SAG and SDG.

This STSM aims not directly at formalization, but at bringing the following advances from SAG to SDG: (1) A local choice axiom. This would allow choices in the sense of sections to surjections on a cover. The right notion of cover has to be found, such that the axiom can be proven to hold in a model. (2) Definition of formally étale maps. In SAG, this definition relies on spaces which are smaller than a point. These spaces are propositions in the sense of homotopy type theory and are not in scope of the usual Kock-Lawvere axioms of SDG, which we therefore aim to extend in their scope.

Working Plan

Description of the work to be carried out by the applicant.

(max.500 word)

Goals (1) and (2) are independent. In both cases, new axioms have to be shown to hold in a model of type theory. It is very reasonable to expect that in SDG, an approach similar to section 8 in [1], which is laid out in more generality in the draft [2] will work. There, a presheaf model is used to internally verify that the modal types for a suitable sheafification modality validate the axioms. The applicant has experience with this kind of modal/synthetic reasoning and will work on this step.

[1] This form is part of the application for a grant to visit a host organisation located in a different country than the country of affiliation. It is submitted to the COST Action MC via-e-COST. The Grant Awarding Coordinator coordinates the evaluation on behalf of the Action MC and informs the Grant Holder of the result of the evaluation for issuing the Grant Letter.

Details about the various models of SDG where laid out by Ulrik Buchholtz at the last SAG meeting [3]. There, the problem of finding the right indexing types for covers in SDG was already briefly discussed. There are reasonable candidates like countable decidable types which might be combined with (synthetic) topological properties like overtness. It should be possible to narrow down the choice of indexing sets in two ways:

(1.a) By application. In SAG, the local choice axiom was found by attempts to synthetically compute cohomology groups. So the applicant will check candidates for local choice axioms by evaluating their use in cohomological computations.

(1.b) By proving that the axiom holds as described above. Since most of the work can be done internally, this should be an economical way of testing.

For goal (2), we can use a similar strategy.

(2.a) A good application should be, among reproduction of the basic proofs in [4], the statement that a formally étale map between manifolds is a local diffeomorphism.

(2.b) Prove that a suitable extension of the Kock-Lawvere axiom holds.

[1]: https://arxiv.org/abs/2307.00073

[2]: https://felix-cherubini.de/sheaves.pdf

[3]:https://www.youtube.com/watch?v=od20xTcENJc&list=PLrnCInSNK7UQ8gBQRsV5WhnMePHvar52y&index=9

[4]: https://felix-cherubini.de/diffgeo.pdf

Expected outputs and contribution to the Action MoU objectives and deliverables.

Main expected results and their contribution to the progress towards the Action objectives (either research coordination and/or capacity building objectives) and deliverables.

(max.500 words)

This STSM contributes to the action and in particular to type theory based interactive theorem proving in a somewhat unusual way. Interactive theorem proving is commonly thought of as the reproduction of classical mathematical results in machine verifyiable form. With a machine verifyable form as the goal, it can be an advantage to change the presentation of the mathematics at the same time.

Synthetic mathematics like SAG and SDG is a heavily changed presentation and leads to shorter proofs. To give a rough indication, the notion of "scheme" in algebraic geometry is formalized (classically) in about 10,000 lines of lean code [5], while the same notion in SAG only needs a couple of houndred lines of code [6]. This is not a fair comparison, since SAG starts from a different basis and in SAG, only quasi-compact quasi-separable schemes of finite presentation are considered. But on the other hand, the basis needs only to be checked once and the essential difficulties and applications are covered by this restricted notion of scheme.

Since a general axiom of choice and the law of excluded middle do not hold in the models of SDG and SAG, it is important to have a constructively minded proof assistant like agda. Reversely, proof assistants could also help mathematicians not used to constructive reasoning to learn these synthetic theories.

This STSM aims at improving our knowledge of the breadth of the synthetic approach. In particular, positive results will give an improved synthetic approach to differential geometry. This helps with one among other new possibilities for formalization of mathematics currently opening up. Formalization of both SAG and SDG, can make use of the already existing and quite extensive formalization of synthetic

homotopy theory, for which homotopy type theory is commonly advertised.

[5]: https://arxiv.org/abs/2101.02602

[6]:https://github.com/felixwellen/synthetic-geometry/blob/main/SyntheticGeometry/qc-Scheme.lagda.md