EuroProofNet STSM application

Details of the STSM

- Title: **Propositional containers and reductions in computability theory**
- Travel dates: February 25–March 9, 2024 (2 weeks)
- Candidate: Danel Ahman, Institute of Computer Science, University of Tartu, Estonia
- Host: Andrej Bauer, Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

Goals of the STSM

1. Investigate the relationship between instance reductions (a notion of reduction arising in constructive mathematics that subsumes Weihrauch reductions in computability theory) and containers (as they are known in type theory and categorical modeling of data types).
2. Apply this relationship to give a type-theoretic account of resource-aware reduction between computational problems (not necessarily computable ones).
3. Formalize instance reductions qua containers to provide a library of formal mathematics with the potential to further formalize some of the more advanced reductions between mathematical statements known from the theory of Weihrauch degrees.

Working plan

Weihrauch reductions [1] are a central topic in computability theory, in which computational difficulty of various mathematical theorems is gauged via reduction of instances of one problem to another. Recently Andrej Bauer showed that Weihrauch reductions have a natural logical formulation in the context of constructive mathematics, where they are subsumed by the notion of instance reductions [2].

It is natural to ask for a type-theoretic account of instance reductions, as that would allow us to incorporate into constructive type theory a rich body of knowledge about computational difficulty of classical mathematical theorems. In our preliminary work on the topic, we discovered that instance reductions are superficially similar to morphisms between containers of Abbott et al [3]. We were able to make the similarity precise (so far unpublished work): the preorder of instance reductions is equivalent to the preorder reflection of the Kleisli category of the inhabited nondeterminism monad on the subcategory of containers spanned by what we call propositional containers (i.e., containers whose position families land in propositions, not in types).

In the proposed STSM, we will investigate and develop this topic further. There are several possible directions of research and questions to be addressed:

- Investigate the mathematical structure of the category of propositional containers further, as we expect this to give us new insights into a functional refinement of instance reductions.
- Type theory and intuitionistic logic can be related in terms of propositional truncation. Can the correspondence between containers and propositional containers be used to give a similar, but more refined correspondence between “resource-aware” notions of logic? (This is at present a rather vague question which we hope to make more precise.)
- Containers are quite literally expressed as type families in type theory, which means that formalization of instance reductions based on propositional containers should be particularly elegant. We would like to confirm that

As part of the STSM we also plan to formalize the relationship between instance reductions and propositional containers in a type theory-based proof assistant, such as Coq or Agda. Time permitting, we shall also test how well some of the classical reductions between Weihrauch degrees lend themselves to formalization in our setup. To our knowledge, this will be the first formalization of Weihrauch reductions, as well as instance reductions.

[1] Klaus Weihrauch. *The Degrees of Discontinuity of Some Translators Between Representations of the Real Numbers*. ICSI Technical Report TR-92-050, 1992.
[2] Andrej Bauer. *Instance reducibility and Weihrauch degrees*. Logical Methods in Computer Science, August 9, 2022, Volume 18, Issue 3.
[3] Abbott, M., Altenkirch, T., Ghani, N. (2003). *Categories of Containers*. In: Gordon, A.D. (eds) Foundations of Software Science and Computation Structures. FoSSaCS 2003. Lecture Notes in Computer Science, vol 2620. Springer, Berlin, Heidelberg.

Expected outputs and contribution to the Action MoU

The results of the STSM will allow us to frame Weirauch reductions and instance reductions in a type-theoretic framework, thereby making it possible to formalize more easily and abstractly these two topics in computability theory and constructive mathematics.

We also expect to gain new mathematical insights into the theory of containers, which generally play an important role in the type-theoretic and mathematical modeling of parametric data types and polymorphic functions, interacting systems, solutions to the view-update problem in database theory, gradient descent based machine learning algorithms, and many more fields.

The results of the STSM will be presented in internationally recognized research venues, such as conferences and peer-reviewed journals. The developed formalization will be made publicly available as an open-source library.

The proposed research contributes most to the goals of WG6 (Type theory) and WG4 (Libraries of formal proofs).