A Multiverse Type Theory

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- MLTT + Erased Types (Ghost type theory), see next talk.

Introduction to SProp

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We can squash types:

$$\frac{A : \mathrm{Type}}{\|A\| : \mathrm{SProp}} \qquad \frac{x : A}{\mathrm{sq}\, x : \|A\|}$$

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Only $\bot_{\mathrm{SProp}} : \mathrm{SProp}$ enjoys an elimination principle into Type.

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$$\frac{A : \mathrm{Type}}{\mathrm{ex}\, A : \mathrm{Exc}} \qquad \frac{a : A}{\mathrm{pure}\, a : \mathrm{ex}\, A}$$

Elimination into Type from Exc needs to handle raise, with catch!

$$\frac{A, B : \text{Type} \qquad a_{\text{ex}} : \text{ex}\, A \qquad a : A \vdash t : B \qquad e : B}{\text{try}\,\text{pure}\, a \leftarrow a_{\text{ex}} \text{ in } t \text{ catch } e : B}$$

These variations on MLTT often combine multiple “universes” (or sorts) of types that behave differently.

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The isolation is key to keeping nice properties of the system:

$$\mathrm{raise}_{\bot} : \bot_{\mathrm{Exc}}$$

at the exceptional sort but the “pure” sort is still consistent, because eliminating $\bot_{\mathrm{Exc}}$ requires handling raise.

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I : IUniv
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Isolation of $\mathbb{I}$ from the rest of the system is key in Cubical, because provability of formulas in $\mathbb{I}$ is decidable.

This leads to a problem:

You have to choose your meta-theory and stick with it.

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Basically vendor lock-in for type theories!

(a) "Agda"

(b) "Coq"

Figure: Your favorite proof assistants

This talk is more of a survey and rough description of an early work-in-progress to spur discussion.

I don't have any definitive answers to all of these questions.

To support the generality of all these systems: we need a structural framework first.

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Some options:

- go PTS-style, an unpublished attempt was made with MuTT in Maillard et al., "The Multiverse: Logical Modularity for Proof Assistants", presented at WG6 2 years ago;
- MTT in Gratzer et al., "Multimodal Dependent Type Theory".

MuTT vs. MTT, structurally

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MuTT vs. MTT, structurally

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$\Gamma \vdash a : A$	$\Gamma \vdash a : A @ m$
$\dfrac{\Gamma \vdash A : s}{\Gamma, A \text{ ctx}}$	$\dfrac{\mu : m \to n \qquad \Gamma.\text{🔒}_\mu \vdash A @ n}{\Gamma, (\mu \mid A) \text{ ctx} @ m}$

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Possible hope of a translation of the structural rules of MuTT to MTT.

Both share some shortcomings:

- No parametrization for `SProp` using them (cannot require *ad-hoc* definitional equalities);

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- MLTT at every sort/mode.

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- Full MTT: Stassen, Gratzer, and Birkedal, “{mitten}: A Flexible Multimodal Proof Assistant”;
- MTT with only one mode and predetermined modalities: Agda!
- Coq’s implementation of Type, Prop and SProp inspired MuTT.

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Inductives in MTT lead to new questions:

- Do inductives exist at all modes?
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- Do inductives exist at all modes?
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- Can't match on $\mathbb{B}$ in SProp to produce a value in Type, because $\mathrm{true} \equiv \mathrm{false}$ would definitionally collapse both elimination branches;
- Have to take care of the raise term when eliminating the exceptional $\mathbb{B}$ into Type.

The case of identity types

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What if instead we had equalities all living at one mode, say SProp?

This is part of the approach of Pujet and Tabareau, “Observational Equality: Now for Good”.

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Key for some systems, for example Cubical!

For others, less so:

- Good luck using $U_{\mathrm{SProp}} @ \mathrm{SProp}$, we want it at Type;
- We might want $U_{\mathrm{ex}} @ \mathrm{ex}$ but also have one at Type.

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Note here that Coq already does this with sort polymorphism, see Pierre-Marie Pédrot's talk at TYPES 2023!

On the MTT side, there's also Andreas' talk at TYPES 2023.

Metatheoretical properties become more elusive:

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What is consistency when $\mathrm{raise}_{\bot} : \bot_{\mathrm{Exc}}$?

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- Consistency at Type in our new system;
- More generally, conservativity for usual MLTT terms and types in Type (a form of relative canonicity).

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- Decidability of typing, maybe even only at certain sorts (if that even makes sense).

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Example: Coq users have had to re-adapt common tactics to work with SProp.

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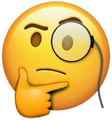

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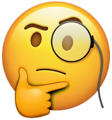

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How could we justify this?

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Roughly what Uemura's recent work attempted to do, presented at WG6 last year.

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Finally, we want some variation on these systems and combinations thereof to be implemented by Coq/Agda.

Thank you for your attention!

Any questions/ideas?