The shape of contexts

Jonas Frey

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Acknowledgements

This work is inspired by a talk by Per Martin-Löf at the 2014 **Workshop on Constructive mathematics and models of type theory**[1] at the **Institut Henri Poincaré** in Paris.

The ideas were developed in discussions with Mathieu Anel, Carlo Angiuli, Chaitanya Leena Subramaniam, and Andrew Swan.

[1] https://ihp2014.pps.univ-paris-diderot.fr/doku.php?id=workshop_2

Contexts in simple and dependent type theory

Contexts in **simple type theory** are flat:

$x_1 : A_1, \dots, x : nA_n \ \vdash \ t(x_1, \dots, x_n) : B$

Contexts in **dependent type theory** are linearly ordered by dependency

$x_1{:}A_1, x_2{:}A_2(x_1), \dots, x_n{:}A(x_1, \dots, x_{n-1}) \vdash B(\vec{x})$

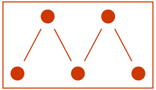
. . . but are they really?

The GAT of categories

Consider the **generalized algebraic theory** $\mathbb{T}_{\mathsf{Cat}}$ of categories:

$$
\begin{aligned}
&\vdash O \\
x\,y : O &\vdash A(x,y) \\
x : O &\vdash \mathrm{id}(x) : A(x,x) \\
x\,y\,z : O\,,\ f : A(x,y)\,,\ g : A(y,z) &\vdash g \circ f : A(x,z) \\
x\,y : O\,,\ f : A(x,y) &\vdash \mathrm{id}(y) \circ f = f \\
x\,y : O\,,\ f : A(x,y) &\vdash f \circ \mathrm{id}(x) = f \\
w\,x\,y\,z : O\,,\ e : A(w,x)\,,\ f : A(x,y)\,,\ g : A(y,z) &\vdash (g \circ f) \circ e = g \circ (f \circ e)
\end{aligned}
$$

The context of $A(x,y)$ has the shape 

The context of composition $g \circ f$ has the shape 

So maybe finite posets are a more realistic representation of dependent contexts than linear orders?

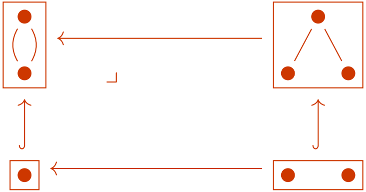
— It turns out posets not enough!

The need for non-posetal shapes

Consider the following **pullback** square in the **syntactic category** $\mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]$ of the GAT $\mathbb{T}_{\mathsf{Cat}}$.

$$\begin{array}{ccc} (x : O\,,\, f : A(x,x)) & \longrightarrow & (x\,y : O\,,\, f : A(x,y)) \\ \downarrow & \lrcorner & \downarrow \\ (x : O) & \longrightarrow & (x\,y : O) \end{array}$$

This pullback lives contravariantly over the following **pushout** of shapes:



Taking the pushout in **posets** doesn't give a well-behaved theory, we have to take it in **categories**.

More precisely in the following category of **finite direct categories**.

Finite direct categories

Definition

1. A category $\mathbb{C}$ is called **direct** if there are no infinite inverse paths $A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \ldots$ of non-identity arrows.
2. A category is called **one-way**, if the only endomorphisms are identities.

Lemma

1. *Direct categories are one-way and skeletal.*
2. *A finite category is direct iff it is one-way and skeletal.*

Definition

FDC is the category of **finite direct categories** and **discrete fibrations**.

FDC *as a coclan*

Among the discrete fibrations, the **injective** ones (a.k.a. **sieve inclusions**) are of special importance: they correspond contravariantly to **context extensions**.

Injective discrete fibrations are closed under composition and pullback (along arbitrary maps) in FDC, and the the initial inclusions $\varnothing \hookrightarrow D$ are obviously injective.

This means that FDC is a **coclan** (dual to a **clan**) with sieve inclusions as **codisplay maps**.

GATs as monads over type structures

A **model** of a coclan $\mathbb{C}$ is a functor $F : \mathbb{C}^{\mathsf{op}} \to \mathsf{Set}$ which sends 0 to 1 and codisplay-pushouts to pullbacks.

Idea

- Models of FDC can be viewed as **context structures** — i.e. the syntactic category of a GAT corresponding only of sort declarations.
- GATs should be certain monads in bimodules over these context structures, in analogy with **algebraic theories** as monads in a Kleisli category of Prof[2].

It is unclear whether all GATs can be represented in this way, since it means reordering the axioms to have sort declarations first.

[2] M. Fiore, N. Gambino, M. Hyland, and G. Winskel. "The Cartesian closed bicategory of generalised species of structures". English. In: *Journal of the London Mathematical Society. Second Series* (2008)

Models of FDC

There is another interpretations of models of FDC which is closer to ideas from Chaitanya's thesis[3]:

Definition

A **locally finite direct category** is a small category $\mathbb{C}$ all of whose slices $\mathbb{C}/c$ are (equivalent to) finite direct categories.

LFDC is the category of locally finite direct categories and discrete fibrations.

For every LFDC $\mathbb{C}$, we can define a functor

$$\mathbb{C} \to \mathsf{FDC}, \qquad c \mapsto \mathbb{C}/c$$

and this functor is a (Street) fibration of groupoids.

The models of FDC are those LFDCs where the groupoids in this fibration are 0-truncated.

(Thanks to Simon Henry for pointing out that the Set-models of FDC do not comprise all LFDCs.)

[3] C. Leena Subramaniam. "From dependent type theory to higher algebraic structures". In: (Oct. 2021). arXiv: 2110.02804 [math.CT].

LFDCs vs DLFCs

In his thesis, Chaitanya considers **direct locally finite categories** (DLFCs). These are the 0-**extensions** in LFDC.

Examples of LFDCs that are not direct:

- The **index category of symmetric graphs** $0 \rightrightarrows 1 \circlearrowleft$ (with an involution on 1) is locally direct but not direct.
- The **terminal LFDC** is the category FDC_0 of finite direct categories with a terminal object.

  It is locally finite direct since we have $\mathsf{FDC}_0/\mathbb{C} = \mathbb{C}$, but not direct, since direct categories may have automorphisms (i.e. Λ).

Since LFDCs are discretely fibered over FDC_0, it turns out that $\mathsf{LFDC} = \widehat{\mathsf{FDC}_0}$ is a **presheaf topos**!

This topos is **étale-subterminal**, in the sense that every other topos admits at most one étale geometric morphism to it.

GATs with well-defined shapes of contexts

In a general GAT, the shape of a context may not be well defined, since contexts of different shapes may be identified by definitional equality.

Preservation of shapes by definitional equality seems to be a kind of **linearity condition**.

I expect this to be related to ideas by Chaitanya on linear GATs.

Thank you for your attention!