Transports

From rewrite rules to axioms

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Problem

- Theory considered: theories encoded in the Calculus of constructions with $\mathcal{T} = (\Sigma_{\mathcal{T}} \cup \Sigma_{CoC}, \mathcal{R}_{\mathcal{T}} \cup \mathcal{R}_{CoC})$

$$Set : \texttt{TYPE} \qquad o : Set \qquad El : Set \to \texttt{TYPE} \qquad Prf : El\ o \to \texttt{TYPE}$$

$$El\ (x \rightsquigarrow_d y) \hookrightarrow \Pi z : El\ x.\ El\ (y\ z)$$

$$Prf\ (x \Rightarrow_d y) \hookrightarrow \Pi z : Prf\ x.\ Prf\ (y\ z)$$

$$El\ (\pi\ x\ y) \hookrightarrow \Pi z : Prf\ x.\ El\ (y\ z)$$

$$Prf\ (\forall\ x\ y) \hookrightarrow \Pi z : El\ x.\ Prf\ (y\ z)$$

- Goal: Replace the rewrite rules $\mathcal{R}_{\mathcal{T}}$ by axioms

Idea

- Replace rewrite rule $\ell \hookrightarrow r$ by equality $\ell = r$
- For every $A \equiv_{\beta\mathcal{R}} B$, build an equality $A = B$
- Given $\Gamma \vdash t : A$ and $\Gamma \vdash p : A = B$, build a transport $\Gamma \vdash \text{transp } p\ t : B$
- Replace each use of the Conversion rule by the insertion of a transport

$\Rightarrow$ Translating the terms = inserting transports

Challenge

- We cannot build an equality $=$ between types because $= : \mathtt{TYPE} \to \mathtt{TYPE} \to \mathtt{TYPE}$ is forbidden

- Restriction
  - $Prf\ a = Prf\ b$ does not exist but $a = b$ does
  - $\Pi x : Set.\ El\ a \to Set = \Pi x : Set.\ El\ b \to Set$ does not exist but $\Pi x : Set.\ (a = b)$ does
  - $\Pi x : El\ a_1.\ El\ a_2 = \Pi x : El\ b_1.\ El\ b_2$ does not exist but $(a_1 \rightsquigarrow_d (\lambda x : El\ a_1.\ a_2)) = (b_1 \rightsquigarrow_d (\lambda x : El\ b_1.\ b_2))$ does

Small types

- Grammars

$$\mathcal{S} ::= Set \mid \mathcal{S} \to Set \mid Set \to \mathcal{S}$$

$$\mathcal{P} ::= Prf\ \ a \mid \mathcal{P} \to \mathcal{S} \mid \Pi z : \mathcal{S}.\ \mathcal{P}$$

$$\mathcal{E} ::= El\ \ b \mid \mathcal{E} \to \mathcal{S} \mid \Pi z : \mathcal{S}.\ \mathcal{E}$$

- A is a small type when $A \equiv_{\beta \mathcal{R}_{CoC}} A'$ with $A' \in \mathcal{S} \cup \mathcal{P} \cup \mathcal{E}$

- If t has type A with A and B small types such that we have a small equality p between A and B, then we can build transp p t of type B

Type system with typed conversion

$$\frac{\Gamma \vdash t : A \qquad (\Gamma \vdash A : s) \equiv (\Gamma \vdash B : s)}{\Gamma \vdash t : B}\ [\textsc{Conv}]$$

$$\frac{\begin{array}{c}(\Gamma_1 \vdash t_1 : \Pi x : A_1.\ B_1) \equiv (\Gamma_2 \vdash t_2 : \Pi x : A_2.\ B_2) \\ (\Gamma_1 \vdash u_1 : A_1) \equiv (\Gamma_2 \vdash u_2 : A_2)\end{array}}{(\Gamma_1 \vdash t_1\ u_1 : B_1[x \mapsto u_1]) \equiv (\Gamma_2 \vdash t_2\ u_2 : B_2[x \mapsto u_2])}\ [\textsc{ConvApp}]$$

Conclusion

- Replacement of the user-defined rewrite rules by axioms
- Restriction to theories encoded in the Calculus of constructions with small types
- Lot of technical details, but the principle remains more or less the same than in Théo's work